Integer solutions for $a^2/1 + (a+1)^2/2 + (a+2)^2/3 + ... + (a+n-1)^2/n = b^2$

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In this paper we consider Diophantine equations of the type $a^2/1 \ + \ (a+1)^2/2 \ + \ (a+2)^2/3 \ + \ ... \ + \ (a+n-1)^2/n \ = \ b^2$. Examples of solutions are given. We give parametric solutions.

I. INTRODUCTION

A few examples of solutions to

$$\frac{a^2}{1} \ + \ \frac{(a+1)^2}{2} \ + \ \frac{(a+2)^2}{3} \ + \ ... \ + \ \frac{(a+n-1)^2}{n} \ = \ b^2 \tag{1}$$

$$\begin{aligned}
&\frac{7^2}{1} + \frac{8^2}{2} = 9^2, \\
&\frac{719^2}{1} + \frac{720^2}{2} = 881^2, \\
&\frac{70487^2}{1} + \frac{70488^2}{2} = 86329^2, \\
&\frac{571^2}{1} + \frac{572^2}{2} + \frac{573^2}{3} = 774^2, \\
&\frac{9659515^2}{1} + \frac{9659516^2}{2} + \frac{9659517^2}{3} = 13079046^2.
\end{aligned}$$

II. MAIN RESULTS

A. "Trivial" solutions

If we set $a \ = \ 1$, then

$$\frac{a^2}{1} \ + \ \frac{(a+1)^2}{2} \ + \ \frac{(a+2)^2}{3} \ + \ ... \ + \ \frac{(a+n-1)^2}{n} \ = \ 1 \ + \ 2 \ + \ 3 \ + \ ... \ + \ n \ = \ \frac{n(n+1)}{2},$$

and our equation becomes

$$\frac{n(n+1)}{2} \ = \ b^2, \tag{2}$$

or

$$(2n+1)^2 \ - \ 8b^2 \ = \ 1. \tag{3}$$

This is a Pell equation with respect to $2n \ + \ 1$ and b. The general solution is

$$n \ = \ \frac{1}{2}\left[- \ 1 \ + \ \frac{1}{2}\Big((3 \ - \ 2\sqrt{2})^k \ + \ (3 \ + \ 2\sqrt{2})^k\Big)\right] \tag{4}$$

The first few solutions are $n \ = \ 1, \ 8, \ 49, \ 288, \ 1681, \ 9800, \ ...$
This already assures that there is an infinite number of solutions to the equation (1).

B. Parametric solutions

For every n, the sum is a quadratic polynomial, so the equation may be reduced to a Pell-like equation. Some of them have solutions.
The two term sum reduces to a Pell-like equation

$$(3a+1)^2 - 6b^2 = -2 \tag{5}$$

The solutions are

$$\begin{aligned} a &= \frac{1}{3}\left[\frac{1}{2}\left((2+\sqrt{6})(-(5-2\sqrt{6})^{2k}) + (\sqrt{6}-2)(5+2\sqrt{6})^{2k}\right) - 1\right] \\ b &= \frac{1}{6}\left[(3+\sqrt{6})(5-2\sqrt{6})^{2k} - (\sqrt{6}-3)(5+2\sqrt{6})^{2k}\right] \end{aligned} \tag{6}$$

The first few solutions are $(a=7,\ b=9),\ (a=719,\ b=881),\ (a=70487,\ b=86329)$.
The three term sum reduces to a Pell-like equation

$$(11a+7)^2 - 66b^2 = -72 \tag{7}$$

The solutions are

$$\begin{aligned} a &= \frac{1}{11}\left[-7 - 3(65-8\sqrt{66})^{2k}(8+\sqrt{66}) + 3(-8+\sqrt{66})(65+8\sqrt{66})^{2k}\right] \\ b &= \left[3 + 4\sqrt{\frac{6}{11}}(65-8\sqrt{66})^{2k} + \frac{1}{11}(33-4\sqrt{66})(65+8\sqrt{66})^{2k}\right] \end{aligned} \tag{8}$$

The first few solutions are $(a=571,\ b=774),\ (a=9659515,\ b=13079046),\ (a=163226494651,\ b=221009718534)$.
Four term equation reduces to a Pell-like

$$u^2 \ - \ 1200b^2 \ = \ -4900, \tag{9}$$

where $u \ = \ 50a+46$. No solutions.
Five term equation reduces to a Pell-like

$$u^2 \ - \ 32880b^2 \ = \ -133200, \tag{10}$$

where $u \ = \ 274a+326$. No solutions.
Are there solutions with larger n? In fact, we know that there are - what we call "trivial solutions" - $n \ = \ 8,\ 49,\ 288,\ ...$
They are parts of solutions of the corresponding Pell-like equations, so there is possibly an infinite set of solutions for the Pell-like equation, which doesn't mean that they are all solutions for an original equation.
$n \ = \ 8$. The Pell-like equation is

$$u^2 \ - \ 852320b^2 \ = \ -10613120, \tag{11}$$

where $u \ = \ 2761a+2958$ There are a few sets of solutions like

$$\begin{aligned} a = \frac{1}{852320}&(-1656480 - 2240(-1572271107665059060(28200116971482675945240048654456 1 \\ &-12218265593682254461308195952 44\sqrt{53270})^k + 6812179537464753 sqrt 53270 \\ &(28200116971482675945240048654456 1 - 12218265593682254461308195952 44\sqrt{53270})^k \\ &-1572271107665059060(28200116971482675945240048654456 1 + 12218265593682254461308195952 44 \\ &\sqrt{53270})^k - 6812179537464753\sqrt{53270}(28200116971482675945240048654456 1 \\ &+12218265593682254461308195952 44\sqrt{53270})^k)) \end{aligned} \tag{12}$$

(produced by Mathematica)
No solutions for n = 9 to 22 - as far as we checked. We expect there will be sets of solutions for other "trivial solutions" - $n \ = \ 49,\ 288$, etc.

III. CONCLUSIONS

In this work we studied Diophantine equations of the type $a^2/1 + (a{+}1)^2/2 + (a{+}2)^2/3 + ... + (a{+}n{-}1)^2/n \ = \ b^2$, Examples of solutions, as well as parametric solutions were given.

Thanks who helped...

References:

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